Electromagnetic waves propagation in nonlinear media with vector field structure considering

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***Abstract*—The paper considers the problem of signal shape transformation during the wave interaction with a continuous nonlinear medium. The proposed method, based on varying the characteristics of a wave in a linear medium, does not lead to the appearance of secular terms, thereby providing a uniform approximation to the exact solution. It was found that during the propagation of the wave, the change in the time delay of the signal is most significant. The influence of the nonlinear medium on the amplitude characteristics of the components of the electromagnetic field is a magnitude of the second order of smallness, and in the first approximation this influence can be neglected. The results are of great fundamental and applied importance for the creation of radiation with desired characteristics (in particular, the expansion of the signal spectrum) and the study of the structure of various media based on nonlinear effects (for example, pathologies of biological tissues at an early stage of their formation).**

***Keywords*—nonlinear media, wave propagation, parameter variation method.**

I. Introduction

The study of nonlinear media is an inevitable stage in the development of science. Every medium exhibits the nonlinear properties when exposed to a certain influence. In particular, the saturation effect is nonlinear, when at high input signal levels the system, due to the limited number of charge carriers, can no longer generate a proportional input signal. At extremely low levels of exposure, it is important to take into account the internal electromagnetic field, which also disrupts the linearity of the medium response. Therefore, at a qualitative level, the environment can be considered as linear only under the condition $E/E_0 << 1$, where E, E_0 are characteristic values of external and internal fields [1]. In reality, however, more complicated situation occurs.

Due to the development of technical possibilities for the creation of generators of high-power and ultra-wideband radiation, the tasks of describing the propagation of radiation in a continuous nonlinear medium have become relevant. Among them, we highlight the problem of using the effects that appear during the interaction of radiation with biological tissues for the timely diagnosis of various diseases. [2,3]. Another class of inverse problems is the problem of signal optimization due to enrichment of its spectrum during propagation in a nonlinear medium, which is still limited to the case of location of nonlinear inclusions [4].

The analysis of signal transformation in nonlinear media are reduced to solving nonlinear partial differential equations, the theory of which is practically absent with the exception of the simplest cases [5]. Therefore, when describing processes in a nonlinear medium, a specific analogue of the complex amplitude method dominates. It is based on an estimate of the signal parameters at frequencies multiple of the fundamental harmonic, which are selected by the type of nonlinear characteristic and are based on physical representations [1]. The imperfection of this approach is visible, for example, in the case of irradiation of a semiconductor diode [6]. Most of the existing approaches are an attempt to solve homogeneous equations, i.e., a description of the possible types of oscillations, the specific combination of which is determined by the source type. This interpretation is possible for linear equations, but in the case of nonlinear media, for which the superposition principle is not valid, it raises serious doubts.

In this paper, an attempt is made to study the problem of the propagation of radiation from a dipole source in a nonlinear medium, the solution of which is free from the above-mentioned shortcomings.

II. Linear medium case

Let us consider the problem of electromagnetic wave propagation in a nonlinear medium, taking into account the vector structure of the electric and magnetic fields, as well as the most general dependence on three spatial and temporal variables ($\vec{r}$ and t). We will start from the system of Maxwell's equations for a homogeneous non-conducting medium in differential form, which have the form

$$\begin{aligned} &\operatorname{rot}\vec{H} = \varepsilon_a \frac{\partial \vec{E}}{\partial t} + \vec{j}, \quad \operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_a}, \\ &\operatorname{rot}\vec{E} = -\mu_a \frac{\partial \vec{H}}{\partial t}, \ \operatorname{div}\vec{H} = 0, \end{aligned} \qquad (1)$$

where are the densities of the external current $\vec{j}$ and charge ρ characterize the sources of the field, and absolute dielectric and magnetic permittivities ε_a, μ_a clearly do not depend on $\vec{r}$ and t. Before considering the solution of

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Maxwell's equations taking into account the nonlinear properties of the medium, let us dwell on the case of a linear non-conductive medium. As a radiation source, we choose a dipole with a constant current distribution along the coordinate z and arbitrary time dependence $\varphi(t)$,

$$j_z = \varphi(t)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\,. \tag{2}$$

We consider that there are no fields, currents and charges at the initial moment of time. We will look for a solution to system (1) by the integral transform method, applying to the equations the Laplace transform on the time variable and the three-dimensional Fourier transform on spatial variables in the Cartesian coordinate system. For further calculations, it is convenient instead of the components of the electric field strength vector $\vec{E}(\vec{r},t)$ consider the components of the electric induction vector $\vec{D}(\vec{r},t)$, $\vec{D}(\vec{r},t) = \varepsilon_a \vec{E}(\vec{r},t)$. The solution of system (1) is

$$\begin{aligned}
&H_x(\vec{r},t) = \frac{\partial}{\partial y}\left[\frac{\varphi\left(t - \frac{R}{v}\right)}{4\pi R}\chi\left(t - \frac{R}{v}\right)\right], \quad H_z(\vec{r},t) = 0\,,\\
&H_y(\vec{r},t) = -\frac{\partial}{\partial x}\left[\frac{\varphi\left(t - \frac{R}{v}\right)}{4\pi R}\chi\left(t - \frac{R}{v}\right)\right],\\
&D_x(\vec{r},t) = \frac{\partial^2}{\partial x \partial z}\left[\frac{\Phi\left(t - \frac{R}{v}\right)}{4\pi R}\chi\left(t - \frac{R}{v}\right)\right],\\
&D_y(\vec{r},t) = \frac{\partial^2}{\partial y \partial z}\left[\frac{\Phi\left(t - \frac{R}{v}\right)}{4\pi R}\chi\left(t - \frac{R}{v}\right)\right],\\
&D_z(\vec{r},t) = \frac{\partial^2}{\partial z^2}\left[\frac{\Phi\left(t - \frac{R}{v}\right)}{4\pi R}\chi\left(t - \frac{R}{v}\right)\right] -\\
&-\mu_a\varepsilon_a\frac{\partial}{\partial t}\left[\frac{\varphi\left(t - \frac{R}{v}\right)}{4\pi R}\chi\left(t - \frac{R}{v}\right)\right],
\end{aligned} \tag{3}$$

where $v = \frac{1}{\sqrt{\varepsilon_a \mu_a}}$ is wave propagation velocity in linear medium, $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$, $\Phi(t) = \int\limits_0^t \varphi(\tau)d\tau$, χ – Heaviside step function.

Summing up the squares of the components of the vectors $\vec{D}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$, we write the expression for the energy density in the spherical coordinate system (up to multiplier $1/2\varepsilon_a$)

$$\begin{aligned}
&D_x^2 + D_y^2 + D_z^2 + \frac{1}{v^2}\left(H_x^2 + H_y^2 + H_z^2\right) =\\
&= \frac{\chi}{(4\pi R)^2}\left\{\sin^2\theta\left[\left(\frac{\varphi'}{v^2} + \frac{\varphi}{vR} + \frac{\Phi}{R^2}\right)^2 +\right.\right.\\
&\left.\left. + \left(\frac{\varphi'}{v^2} + \frac{\varphi}{vR}\right)^2 + 4\cos^2\theta\left(\frac{\varphi}{vR} + \frac{\Phi}{R^2}\right)^2\right]\right\}.
\end{aligned}$$

Here all functions have the argument $(t - R/v)$ omitted, and the prime means the derivative with respect to the entire argument.

III. PARAMETERS VARIATION METHOD

To solve Maxwell's equations in the case of a nonlinear medium, the parameters variation method is applicable [7]. As the initial form we choose solution (3) of system (1). We will vary the amplitude and time delay for the field components and introduce additive components into consideration. As a result, the solution to the equations system (1) for a nonlinear dependence of the permittivity on the applied field magnitude will be sought in the form

$$\begin{aligned}
&H_x(\vec{r},t) = \frac{\partial}{\partial y}\left[\left(m_1(\vec{r},t)\varphi(\tau(\vec{r},t)) + s_1(\vec{r},t)\right)\chi(\tau(\vec{r},t))\right],\\
&H_z(\vec{r},t) = 0\,,\\
&H_y(\vec{r},t) = -\frac{\partial}{\partial x}\left[\left(m_2(\vec{r},t)\varphi(\tau(\vec{r},t)) + s_2(\vec{r},t)\right)\chi(\tau(\vec{r},t))\right],\\
&D_x(\vec{r},t) = \frac{\partial^2}{\partial x \partial z}\left[\left(m_4(\vec{r},t)\varphi(\tau(\vec{r},t)) + s_4(\vec{r},t)\right)\chi(\tau(\vec{r},t))\right],\\
&D_y(\vec{r},t) = \frac{\partial^2}{\partial y \partial z}\left[\left(m_5(\vec{r},t)\varphi(\tau(\vec{r},t)) + s_5(\vec{r},t)\right)\chi(\tau(\vec{r},t))\right],\\
&D_z(\vec{r},t) = \frac{\partial^2}{\partial z^2}\left[\left(m_6(\vec{r},t)\varphi(\tau(\vec{r},t)) + s_6(\vec{r},t)\right)\chi(\tau(\vec{r},t))\right] -\\
&-\mu_a\varepsilon_a\frac{\partial}{\partial t}\left[\left(m_6(\vec{r},t)\varphi(\tau(\vec{r},t)) + s_6(\vec{r},t)\right)\chi(\tau(\vec{r},t))\right].
\end{aligned} \tag{4}$$

The functions $m_{1,2,4,5,6}(\vec{r},t)$ have the physical meaning as the coefficient of wave transmission in a nonlinear medium. The function $\tau(\vec{r},t)$ describes the wave delay during propagation due to a change in its phase velocity by a nonlinear medium, it can be related to the equivalent refractive index $n(\vec{r},t)$ for a nonlinear medium by the relation

$$\tau(\vec{r},t) = t - \frac{|\vec{r}|}{v}n(\vec{r},t)\,. \tag{5}$$

The functions $s_{1,2,4,5,6}(\vec{r},t)$ represent secondary radiation, the propagation direction of which does not coincide with the primary wave. We emphasize that the relationship between these functions in (4) is quite arbitrary, but reflects the basic ideas about changes in wave characteristics during propagation in a nonlinear homogeneous medium and hopes for an adequate description of processes in a nonlinear medium.

To find unknown functions $m_{1,2,4,5,6}(x,t)$, $\tau(x,t)$ and $s_{1,2,4,5,6}(x,t)$, it is necessary to substitute expressions (4) into system (1), written out for each component, where the permittivity will depend on the magnitude of the applied field, and the dipole (2) is the source of the field, as in the case of a linear medium.

To simplify expressions, we will omit the arguments of the functions $\Phi(\tau(\vec{r},t))$, $\varphi(\tau(\vec{r},t))$, $\chi(\tau(\vec{r},t))$, $\delta(\tau(\vec{r},t))$, $m_{1,2,4,5,6}(\vec{r},t)$, $s_{1,2,4,5,6}(\vec{r},t)$, $\tau(\vec{r},t)$ and use subscripts to denote the corresponding derivatives, for example, $\varphi_x \equiv \dfrac{\partial \varphi}{\partial x}$, $m_{tt} \equiv \dfrac{\partial^2 m}{\partial t^2}$. Designation φ' means a derivative over the entire argument. For example, write the first equation of system (1) after substitution

$$
\begin{gathered}
\left(m_{2z}\varphi + m_{2z}\varphi'\tau_x + m_{2x}\varphi'\tau_z + m_2\varphi''\tau_x\tau_z + m_2\varphi'\tau_{xz} + s_{2xz}\right)\chi + \\
+\left(m_{2z}\varphi + m_2\varphi'\tau_z + s_{2z}\right)\tau_x\delta + \left(m_{2x}\varphi + m_2\varphi'\tau_x + s_{2x}\right)\tau_z\delta + \\
+\left(m_2\varphi + s_2\right)\tau_{xz}\delta + \left(m_2\varphi + s_2\right)\tau_z\tau_x\delta' = \\
\left(m_{4xzt}\Phi + m_{4zt}\varphi\tau_x + m_{4xt}\varphi\tau_z + m_{4xz}\varphi\tau_t + m_{4t}\varphi'\tau_x\tau_z + \right. \\
+ m_{4t}\varphi\tau_{xz} + m_{4z}\varphi'\tau_x\tau_t + m_{4z}\varphi\tau_{xt} + m_{4x}\varphi'\tau_t\tau_z + \\
+ m_4\varphi''\tau_x\tau_z\tau_t + m_4\varphi'\tau_{xt}\tau_z + m_4\varphi'\tau_t\tau_{zx} + \\
\left. + m_{4x}\varphi\tau_{zt} + m_4\varphi'\tau_x\tau_{zt} + m_4\varphi\tau_{xzt} + s_{4xzt}\right)\chi + \left(m_{4zt}\Phi\tau_x + \right. \\
+ m_{4t}\varphi\tau_z\tau_x + m_{4z}\varphi\tau_t\tau_x + m_4\varphi'\tau_t\tau_z\tau_x + m_4\varphi\tau_{zt}\tau_x + \\
+ s_{4zt}\tau_x + m_{4x}\varphi\tau_t\tau_z + +m_{4x}\varphi\tau_t\tau_z + m_4\varphi'\tau_x\tau_t\tau_z + \\
\left. + m_4\varphi\tau_{tx}\tau_z + m_4\varphi\tau_t\tau_{tx}\right)\delta + m_4\varphi\tau_t\tau_z\tau_x\delta'.
\end{gathered} \tag{6}
$$

Equating the coefficients at δ', we obtain a system of equations

$$
\begin{aligned}
&\left(m_2\varphi + s_2\right)\tau_z\tau_x = m_4\varphi\tau_t\tau_z\tau_x, \\
&\left(m_1\varphi + s_1\right)\tau_z\tau_y = m_5\varphi\tau_t\tau_z\tau_y, \\
&-\left(m_2\varphi + s_2\right)\tau_x^2 - \left(m_1\varphi + s_1\right)\tau_y^2 = \\
&= m_6\varphi\tau_t\tau_z^2 - \mu_a\varepsilon_a\left(m_6\varphi\tau_t^2 + s_6\tau_t^2\right).
\end{aligned} \tag{7}
$$

Since the exact solution of this system is impossible due to the excess of the number of unknowns over the number of equations, we will solve it assuming a small difference in the signal from the case of a linear medium. Therefore, like the geometric optics method, we will consider phase relations to be priority, and approximately we will take $m_i = 1/4\pi R$ and $s_i = 0$. Then the first two equations of system (7) follow $\tau_t = 1$, and from the latter

$$
\tau_x^2 + \tau_y^2 + \tau_z^2 = \mu_a\varepsilon_a. \tag{8}
$$

The form of this equation coincides with the well-known eikonal equation in the approximation of geometric optics, but an important difference is the dependence ε_a on delay $\tau(\vec{r},t)$. In a spherical coordinate system, this equation would look like

$$
R^2\tau_R^2 + \tau_\theta^2 = R^2\mu_a\varepsilon_a, \tag{9}
$$

where the independence of the function τ from the azimuthal angle φ is taken into account for symmetry reasons, $\tau_\varphi = 0$.

We will consider the case when the nonlinear dependence is determined by the level of energy acting on the medium, i.e.

$$
\begin{gathered}
\mu_a\varepsilon_a = \frac{1}{v^2}\left(1 + \gamma f\left(\frac{\chi}{(4\pi R)^2}\left\{\sin^2\theta\left[\left(\frac{\varphi'}{v^2} + \frac{\varphi}{vR} + \frac{\Phi}{R^2}\right)^2 + \right.\right.\right.\right. \\
\left.\left.\left.\left. + \left(\frac{\varphi'}{v^2} + \frac{\varphi}{vR}\right)^2\right] + 4\cos^2\theta\left(\frac{\varphi}{vR} + \frac{\Phi}{R^2}\right)^2\right\}\right)\right),
\end{gathered} \tag{10}
$$

where v is wave propagation velocity in background linear medium, γ is a small numerical parameter characterizing the influence of a nonlinear characteristic f.

IV. Calculating the function $\tau(\vec{r},t)$

A. Plane wave approximation

Even in the approximate form of equation (8) or (9) with the right-hand side of the form (10), the solution to the problem of the wave phase dependence in a nonlinear medium is apparently impossible. Further methods of simplifying the problem are needed, in which a qualitative analysis of the wave behavior is possible. One of them, describing the propagation of a wave in the far field, is the "plane wave approximation". It consists of neglecting the dependence of the permittivity on the coordinates R, θ, and taking into account only the dependence on the function $\tau(\vec{r},t)$. The rationale for this approximation is the analogy with the propagation of radiation in a linear medium, when at a sufficient distance from the emitter the wave can be considered as plane, neglecting the decrease in amplitude with distance and taking into account only the phase relationships. Then the problem of the first-order partial differential equation (8) solving, where $\varepsilon_a = \varepsilon_0\left(1 + \gamma f\left(\varphi(\tau)\right)\right)$, allows for an exact solution.

Let's make a replacement

$$
u = \int \frac{d\tau}{\sqrt{\varepsilon_a\mu_a}}. \tag{11}
$$

Then the equation is transformed to the form [8] $u_x^2 + u_y^2 + u_z^2 = 1$. For this nonlinear equation with constant coefficients, the complete integral is known

$$
u = C_1x + C_2y + C_3z + C_4,
$$

where $C_1^2 + C_2^2 + C_3^2 = 1$ [8]. However, since the same equation describes the process of radiation propagation in a linear homogeneous medium, the desired particular solution has the form $u = vt - R$. The solution of the original equation (8) is obtained as the solution of the transcendental equation (11) with respect to the variable τ by the found value u.

B. Direct decomposition method

Let us apply the direct decomposition method to equation (9). This method gives correct results only at a small distance from the source, but allows for a more accurate account of the structure of the dependence (10). We will look for a solution in the form of an expansion in a small parameter γ: $\tau = \tau_0 + \gamma\tau_1 + \ldots$. In the zero approximation, corresponding to the case of a linear medium, we have the solution

$$\tau_0 = t - \frac{R}{v},$$
$$R^2\left(\frac{1}{v^2} - \frac{2\gamma}{v}\tau_{1R} + \gamma^2\tau_{1R}^2 + \ldots\right) + \gamma^2\tau_{1\theta}^2 + \ldots =$$
$$= \left(\frac{R}{v}\right)^2 + \frac{\gamma}{v^2}F\chi(\tau_0) + \ldots,$$

where

$$F(R,\theta,\tau_0) = R^2 f\left(\frac{1}{(4\pi R)^2}\left\{\sin^2\theta\left[\left(\frac{\varphi'(\tau_0)}{v^2} + \frac{\varphi(\tau_0)}{vR} + \frac{\Phi(\tau_0)}{R^2}\right)^2 + \right.\right.\right.$$
$$\left.\left.\left. + \left(\frac{\varphi'(\tau_0)}{v^2} + \frac{\varphi(\tau_0)}{vR}\right)^2\right] + 4\cos^2\theta\left(\frac{\varphi(\tau_0)}{vR} + \frac{\Phi(\tau_0)}{R^2}\right)^2\right\}\right).$$

Neglecting terms of the second and higher orders of smallness, we obtain an equation for the first approximation

$$\tau_{1R} = -\frac{F\chi}{2R^2 v}.$$

By integrating this ordinary differential equation, we obtain an expression for the function τ_1

$$\tau_1 = -\frac{1}{2v}\int_0^R \frac{F\chi\left(t - \frac{R}{v}\right)dR}{R^2} = -\frac{1}{2v}\int_0^{vt}\frac{FdR}{R^2}. \qquad (12)$$

C. Pfaff's method

Let us write equation (9) in a form resolved with respect to one of the derivatives

$$\tau_R = -\sqrt{\mu_a\varepsilon_a(R,\theta,\tau) - \frac{1}{R^2}\tau_\theta^2}. \qquad (13)$$

We chose the minus sign by analogy with the linear case, where $\tau = t - \sqrt{\mu_a\varepsilon_a}R$.

Let's apply Pfaff's method [9] to solve this equation. Let us consider the relation for the total differential of an unknown function $d\tau = \tau_\theta d\theta + \tau_R dR$, which, after taking into account (13), takes the form

$$d\tau - \tau_\theta d\theta + \sqrt{\mu_a\varepsilon_a(R,\theta,\tau) - \frac{1}{R^2}\tau_\theta^2}\,dR = 0. \qquad (14)$$

Let's replace all variables, excluding R, through new parameters u_1, u_2, u_3:

$$\theta = \xi(R,u_1,u_2,u_3),\ \tau = \zeta(R,u_1,u_2,u_3),$$
$$\tau_\theta = \eta(R,u_1,u_2,u_3). \qquad (15)$$

When substituting into equation (14), taking into account that

$$d\tau = \frac{\partial\zeta}{\partial R}dR + \frac{\partial\zeta}{\partial u_1}du_1 + \frac{\partial\zeta}{\partial u_2}du_2 + \frac{\partial\zeta}{\partial u_3}du_3,$$

we receive

$$\left(\frac{\partial\zeta}{\partial R} - \eta\frac{\partial\xi}{\partial R} + \sqrt{\mu_a\varepsilon_a(R,\theta,\tau) - (\eta/R)^2}\right)dR +$$
$$+ \sum_{i=1}^{3}\left(\frac{\partial\zeta}{\partial u_i} - \eta\frac{\partial\xi}{\partial u_i}\right)du_i = 0. \qquad (16)$$

There is sufficient arbitrariness in the choice of functions ξ, η, ζ, but they must satisfy the conditions

$$\frac{\partial\zeta}{\partial R} - \eta\frac{\partial\xi}{\partial R} + \sqrt{\mu_a\varepsilon_a(R,\xi,\zeta) - (\eta/R)^2} = 0. \qquad (17)$$

It has been proven [9] that such functions always exist.

We assume that the function ξ corresponding to the polar angle θ does not depend on R. Then, integrating the first condition (17) with respect to the variable R, we obtain the integral equation

$$\zeta = t - \int_0^R \sqrt{\mu_a\varepsilon_a(R,\xi,\zeta) - (\eta/R)^2}\,dR, \qquad (18)$$

which establishes the connection of the function ζ with the other introduced functions. The validity of the established type for function ζ and in fact for the desired function τ confirmed by particular cases. For example, the complete integral of the equation $\tau_R^2 + \frac{1}{R^2}\tau_\theta^2 = f(R)$ has the form [8]

$$\tau(R,\theta)=C_1\theta\pm\int_0^R\sqrt{f(R)-(C_1/R)^2}\,dR+C_2.$$

In addition, the function obtained from equation (18) will satisfy the natural boundary condition $\tau|_{R=0}=t$.

Let's consider the left part of the second condition (17)

$$\frac{\partial\zeta}{\partial u_i}-\eta\frac{\partial\xi}{\partial u_i}=-\int_0^R\frac{dR}{\sqrt{\mu_a\varepsilon_a(R,\xi,\zeta)-(\eta/R)^2}}\cdot$$
$$\cdot\left[\mu_a\left(\frac{\partial\varepsilon_a}{\partial\xi}\frac{\partial\xi}{\partial u_i}+\frac{\partial\varepsilon_a}{\partial\zeta}\frac{\partial\zeta}{\partial u_i}\right)-\frac{1}{R^2}\frac{\partial\eta}{\partial u_i}\right]-\eta\frac{\partial\xi}{\partial u_i}.$$

So that at different u_i these expressions had a common factor describing the dependence on the variable R, need to require the function dependencies ξ,η,ζ by variable $u=u_1+u_2+u_3$. Then

$$M=-\int_0^R\frac{dR}{\sqrt{\mu_a\varepsilon_a-(\eta/R)^2}}\left[\mu_a\left(\frac{\partial\varepsilon_a}{\partial\xi}\frac{\partial\xi}{\partial u}+\frac{\partial\varepsilon_a}{\partial\zeta}\frac{\partial\zeta}{\partial u}\right)-\frac{1}{R^2}\frac{\partial\eta}{\partial u}\right]-$$
$$-\eta\frac{\partial\xi}{\partial u},\ Q_i=1.$$

After substituting these functions into equation (16), we have an extremely simple equation in total differentials $du=0$, the general solution of which $\Psi(u)=0$, where Ψ is arbitrary function. Now, to obtain the general integral, it is necessary to finally decide on the choice of functions $\xi(u),\eta(R,u)$, express the variable u from equations (17) through the original variables R,θ,τ and, substituting into the function Ψ, obtain the general integral of equation (13).

The proposed method for obtaining a solution cannot be fully implemented due to the complex dependence (18) of the function ζ from u. In addition, to specify the function Ψ a boundary condition of the form $\tau|_{\theta=\theta_0}=\psi(R)$ is required. It cannot be found. Therefore, to determine the wave phase characteristic in a nonlinear medium, we use the limiting cases of a linear and quasilinear medium. In this case, for simplicity, we use the first approximation of the integral equation (18), in which in the integrand

$$\zeta=\zeta_0=t-R/v. \qquad (19)$$

For a linear medium, the nonlinear dependence (10) becomes a constant $\mu_a\varepsilon_a=\frac{1}{v^2}$, and in order to obtain dependence (18), it is necessary to demand $\eta|_{\gamma=0}=0$. Comparing (18) with the result obtained in the small perturbation approximation (12)

$$\zeta\approx t-\frac{1}{v}\int_0^R\sqrt{1+\gamma\chi(\zeta_0)f-(\eta v/R)^2}\,dR\approx t-\frac{R}{v}-\frac{\gamma}{2v}\int_0^{vt}f\,dR$$

we are convinced that the contribution of the function η is of the second order of smallness, and one can use the case $\eta\approx 0$.

To find the function $\xi(u)$, we use the last of the equations (15), which can be rewritten

$$\frac{\partial\zeta/\partial u}{\partial\xi/\partial u}=-\int_0^R\frac{dR}{\sqrt{\mu_a\varepsilon_a(R,\xi,\zeta)-(\eta/R)^2}}\cdot$$
$$\cdot\left[\mu_a\left(\frac{\partial\varepsilon_a}{\partial\xi}+\frac{\partial\varepsilon_a}{\partial\zeta}\frac{\partial\zeta}{\partial u}\left(\frac{\partial\xi}{\partial u}\right)^{-1}\right)-\frac{1}{R^2}\frac{\partial\eta}{\partial u}\right]=\eta\left(1+\frac{\partial\xi}{\partial u}\right).$$

Taking into account the approximations made, we can also talk about the insignificance of the effect of mutual dependence of the angular distribution and phase characteristics of the wave in a nonlinear medium. Thus, in the first approximation, the phase characteristic can be calculated using the formula

$$\tau=t-\frac{1}{v}\int_0^R\sqrt{1+\gamma\chi(t-R/v)f}\,dR=t-\int_0^R\sqrt{\varepsilon_a\mu_a}\,dR. \qquad (20)$$

V. Phase dependence modeling and results analysis

The main obstacle to the numerical implementation of various methods for determining the signal delay during the propagation of radiation in a nonlinear medium is its dependence on the magnitude of the field (or its energy density) along the entire propagation path from the source to the receiving point. At the same time, the problem of correctly finding the value of the characteristics of the electromagnetic field near the source is a difficult task associated with the field singularity in point source at $R\to 0$. Thus, the values of the fields (3) in the immediate vicinity of the emitter are incorrect, and a more detailed analysis of the magnitude of these fields is required here. An attempt to solve this problem was presented in [10]. The singularity-free expression for the energy density is

$$W=\frac{1}{2\varepsilon_a}\left[D_x^2+D_y^2+D_z^2+\frac{1}{v^2}\left(H_x^2+H_y^2+H_z^2\right)\right]=$$
$$=\frac{\chi\left(t-\frac{R}{v}\right)I^2\sin^2\theta}{2(4\pi)^2\varepsilon_a}\left\{\left[\left(\frac{\varphi'}{v^2}\right)^2+\left(\frac{\varphi'}{v^2}+\frac{\varphi}{vR}\right)^2\right]\right\}, \qquad (21)$$
$$I\approx\pi a^2\ln\left|\frac{\sqrt{a^2+\rho^2+\left(\frac{h}{2}-z\right)^2}+\frac{h}{2}-z}{\sqrt{a^2+\rho^2+\left(\frac{h}{2}+z\right)^2}-\frac{h}{2}-z}\right|,$$

where h and a are length and the cross-section radius of a small cylindrical radiator corresponding to a dipole (2).

Numerical modeling of the wave phase characteristic and the equivalent refractive medium index (5) was carried out based on expressions (20) and (21). The source of influence was a cylindrical emitter with a time dependence in the Gaussian pulse form

$$\varphi(t) = Ae^{-\beta(t-t_0)^2}\sin\omega(t-t_0),$$

where $A = 0.5$, $\beta = 20$ GHz, $\omega = 2\pi f$, $f = 4$ GHz, $t_0 = 0.13$ ns, $h = 0.05$ m, $a = 0.005$ m, $\gamma = 0.0001$. Fig. 1 shows the increment of the equivalent refractive index $n-1$ from (5) due to the nonlinearity of the medium.

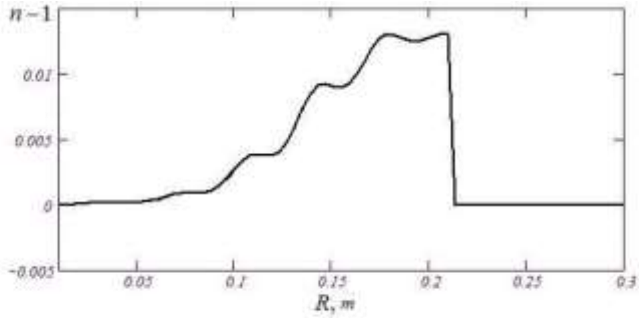

Fig.1. Increment of the equivalent refractive index in the Gaussian pulse case.

The simulation showed that due to the rapid decrease in wave energy density in the medium (according to (21), in the near zone of the emitter its decrease $\sim 1/R^4$) already at a short distance from the emitter, the amplitude of the wave becomes insufficient for the response of the nonlinear medium to the effect. This effect prevents the accumulation of nonlinear effects due to the multiple interaction of the wave with the medium. Therefore, taking into account the sphericity of the wave, the occurrence of shock waves is excluded even in the absence of dispersion. For a mathematical description of the propagation process, we can limit ourselves to single scattering approximations, which generalize the Born approximation.

VI. Amplitude Characteristics of the Electromagnetic Field

Since the direct solution of the system of equations for the amplitude functions $m_i(\vec{r},t)$, $i = 1,2..6$ obtained by equating the coefficients at $\delta(\tau(\vec{r},t))$ in (6) and other Maxwell's system equations, impossible, first of all, due to the extreme bulkiness of its recording, we will undertake certain simplifications. First of all, let's stop trying to find functions $s_i(\vec{r},t)$, characterizing the scattered field, setting them equal to zero. Let us write the system of equations (1) more compactly without directly calculating the derivatives:

$$\frac{\partial^2}{\partial x\partial z}(m_2\varphi\chi) = \frac{\partial^3}{\partial x\partial z\partial t}(m_4\Phi\chi),$$
$$\frac{\partial^2}{\partial y\partial z}(m_1\varphi\chi) = \frac{\partial^3}{\partial y\partial z\partial t}(m_5\Phi\chi),$$
$$\frac{\partial^2}{\partial x^2}(m_2\varphi\chi) + \frac{\partial^2}{\partial y^2}(m_1\varphi\chi) = -\frac{\partial^3}{\partial z^2\partial t}(m_6\Phi\chi) +$$
$$+\frac{1}{v^2}\frac{\partial}{\partial t}\left[(1+\gamma f\chi)\frac{\partial}{\partial t}(m_6\varphi\chi)\right] + \varphi(t)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0),$$
$$\frac{\partial^3}{\partial z^2\partial y}\left[(m_6-m_5)\Phi\chi\right] = \frac{1}{v^2}\frac{\partial}{\partial y}\left[(1+\gamma f\chi)\frac{\partial}{\partial t}(m_6\varphi\chi)\right] -$$
$$-\frac{1}{v^2}(1+\gamma f\chi)\frac{\partial^2}{\partial y\partial t}(m_1\varphi\chi), \quad (19)$$
$$\frac{\partial^3}{\partial z^2\partial x}\left[(m_6-m_4)\Phi\chi\right] = \frac{1}{v^2}\frac{\partial}{\partial x}\left[(1+\gamma f\chi)\frac{\partial}{\partial t}(m_6\varphi\chi)\right] -$$
$$-\frac{1}{v^2}(1+\gamma f\chi)\frac{\partial^2}{\partial x\partial t}(m_2\varphi\chi),$$
$$\frac{\partial^3}{\partial x\partial y\partial z}\left[(m_5-m_4)\Phi\chi\right] = 0.$$

The last equation implies the equality $m_4 = m_5$, and from the first two – $m_1 = m_2$, i.e. the symmetry of the problem with respect to transverse coordinates x, y is physically obvious, therefore, only 3 equations out of 6 will be informative. Moreover, we can equate the arguments of the same derivatives on the left and right, since in the linear case all amplitude functions are equal, and possible additional arbitrary functions ("constants" with respect to the arguments of the derivatives used) will therefore be equal to zero. It is also convenient to switch to a cylindrical coordinate system.

As a result, the system of equations (22) is transformed to the form

$$m_1\varphi = \frac{\partial}{\partial t}(m_4\Phi), \quad (23)$$
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial(m_1\varphi)}{\partial\rho}\right) = -\frac{\partial^3}{\partial z^2\partial t}(m_6\Phi) + \frac{1}{v^2}\frac{\partial}{\partial t}\left[(1+\gamma f\chi)\frac{\partial}{\partial t}(m_6\varphi)\right], \quad (24)$$
$$\frac{\partial^3}{\partial z^2\partial\rho}\left[(m_6-m_4)\Phi\right] = \frac{1}{v^2}\frac{\partial}{\partial\rho}\left[(1+\gamma f\chi)\frac{\partial}{\partial t}(m_6\varphi)\right] -$$
$$-\frac{1}{v^2}(1+\gamma f\chi)\frac{\partial^2}{\partial\rho\partial t}(m_1\varphi). \quad (25)$$

We will solve the system using the method of successive approximations. We will reduce equations (23)-(25) to integral form. Let us single out the linear d'Alembert operator in them. We integrate both parts of (24) with respect to the variable t and take into account $\frac{\partial}{\partial t}(m_6\varphi) = \frac{\partial^2}{\partial t^2}(m_6\Phi) - \frac{\partial}{\partial t}\left(\frac{\partial m_6}{\partial t}\Phi\right)$, then

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)(m_6\Phi) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\int_0^t m_1\varphi\,dt\right) -$$
$$-\frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial m_6}{\partial t}\Phi\right) + \frac{\gamma f\chi}{v^2}\frac{\partial}{\partial t}(m_6\varphi).$$

Likewise

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)(m_4\Phi)=$$
$$=\frac{\partial^2}{\partial z^2}(m_6\Phi)-\frac{1}{v^2}(1+\gamma f\chi)\frac{\partial}{\partial t}(m_6\varphi)+\frac{\gamma f\chi}{v^2}\frac{\partial^2}{\partial t^2}(m_4\Phi)-$$
$$-\frac{\gamma}{v^2}\int_0^\rho \frac{\partial f}{\partial \rho}\frac{\partial^2}{\partial t^2}(m_4\Phi)\chi d\rho.$$

The last terms on the right-hand sides can be neglected as a first approximation, since they represent quantities of the order γ^2. As a result of solving the one-dimensional wave equation, we arrive at a system of integro-differential equations

$$m_1\varphi=\frac{\partial}{\partial t}(m_4\Phi),$$
$$m_4\Phi=m_6\Phi-\frac{1}{2v}\int_0^t d\tau\int_{z-v(t-\tau)}^{z+v(t-\tau)}\frac{\partial}{\partial \tau}\left(\frac{\partial m_6}{\partial \tau}\Phi\right)d\zeta\ ,$$
$$m_6\Phi=\frac{v}{2}\int_0^t d\tau\int_{z-v(t-\tau)}^{z+v(t-\tau)}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\int_0^t m_1\varphi dt\right)+\frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial m_6}{\partial t}\Phi\right)\right]d\zeta,$$

which is supposed to be solved iteratively.

Note that, since the resulting system of equations does not explicitly include a nonlinear characteristic, its solution will lead to functions corresponding to the linear case. It follows that the influence of the nonlinear medium on the amplitude characteristics of the components of the electromagnetic field is a magnitude of the second order of smallness, and in the first approximation this influence can be neglected.

VII. CONCLUSION

The problem setting methodology proposed in this article differs from the traditional one in two aspects: 1) a wave from a specific source in the form of a dipole is investigated; 2) the solution is carried out for a signal with an arbitrary time dependence, as well as the general type of non-linear characteristic of the medium. The solution to this problem and subsequent numerical modeling made it possible to identify the general patterns of waveform transformation by nonlinear scatterers and to analyze the patterns of wave distortion depending on the shape of the original signal and the type of nonlinear characteristic.

The method of variation of field parameters of a linear problem is applied to the solution of a system of Maxwell's equations for a nonlinear medium. The elementarity of the radiation source used makes it possible to determine unknown functions from the condition of equality of coefficients for the δ-function, its derivative and antiderivative.

The nonlinear medium has the greatest influence on the wave phase. Since even in an approximate form the exact solution of this equation, which has an analogy with the eikonal equation of geometric optics, is apparently impossible, several methods of simplifying the problem are considered (the "plane wave approximation", the direct expansion method, the Pfaff method).

Despite the complexity of the problem, which leads to inevitable approximations, the approach used allows us to identify the effects of accumulation of changes in the amplitude, phase and polarization characteristics of a signal during wave propagation in a nonlinear medium. This is what makes it stand out from classical methods based on the use of the small disturbance method, the results of which are valid only in the near zone of the source, where the structure of the influencing signal itself requires correction.

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