Modeling nonlinear evolutionary competing processes on the basis of the Lotka – Volterra modification

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Abstract— **The models of nonlinear competing processes on the base of the modification of the Lotka – Volterra system: the Richardson arms race model, the Lanchester war or battle model, the predator-prey model are presented and studied in the article.**

The qualitative analysis and also numerical calculation of evolutionary trajectories of system for various values of parameters is carried out. The comparative analysis of the classical and modified models on the basis of the built graphs of solutions of the system of differential equations is presented as well.

Various parameters of the system of differential equations which cover practically all the set of possible beneficial and adverse effect on evolutionary trajectories are considered. Due to the dynamic simulation the values and relations of parameters which lead to the cardinal changes in evolution of the modified models are defined. Thus, the conditions under which it is possible to avoid the growth of arms race, that is, the armed conflict, are received. For the modified predator-prey model the conditions under which the biocenosis restoration is possible even at negative nonlinear terms are written down.

***Keywords*—Lanchester war or battle model, Lotka – Volterra model, modified Lanchester model, modified predator-prey model, Richardson arms race model.**

I. Introduction

The history of mankind is penetrated by wars and armed conflicts (an armed conflict is characterized, unlike a war, by the limitation of purposes). In the modern world the news blocks are also devoted to describing wars and armed conflicts both in a certain state and between states. The main struggle between states is deployed for economic and political hegemony. In this regard the arms race growths and we know that the modern arms are closely connected with the development of science. The systems of differential equations describing real, absolutely various phenomena in physics, mechanics, policy, biology and other areas can be closely connected among themselves. Their evolution is often based on the competing processes. In this work we consider the following phenomena: the arms race on the basis of Richardson's model; the war or battle on the basis of Lanchester's model; the dynamics of the predator-prey system. Since the evolution of the studied models is defined by the competition of opposite processes, in the article we consider the Lotka – Volterra system [1, 2, 3] as the base of mathematical modeling.

The problem under investigation is of great interest since the qualitative and numerical analysis of models of real processes defines not only the evolutionary tendency of systems of differential equations, but also the result: an infinite arms race or a mutual disarmament; victory or defeat in war; the balanced existence of a biocenosis or formation of dead zones.

There are various models of biocenosis evolution: Lotka – Volterra; Kolmogorov; Holling – Tanner; Rosenzweig – MacArthur, etc. In this work we consider the modified Lotka – Volterra model .

The mathematical Lotka – Volterra model, also it is called the predator-prey model, is applied to describing evolutionary processes in various areas of human activity, in particular, in biology, ecology, economy, sociology, medicine, history, computer informatics. Let us present several examples: the model of military conflicts; the model of the class struggle; the model of spreading epidemics; the model of equalizing prices of goods, the model of infection with a computer virus. Indeed, on the basis of the Lotka – Volterra model it is possible to consider all models where competitive relationships between the studied objects exist.

The presented model is set by the system of the differential equations describing the competing processes in which we use the parameters but not functional dependences of coefficients. Thanks to the computer modeling we can consider a large number of values of parameters and their combinations. Thus, it is obvious the influence of external and internal factors on an ecosystem, it is also possible to allocate their negative and positive effect on the evolution.

Let us note that in the article the real concrete historical events are not studied since mathematical models are always restricted: it is impossible to take into account random factors which are connected with stochastic processes and have a probabilistic character [4, 5], however, if random factors are negligible, then the obtained results define evolutionary trajectories of historical events. The studied modified models are deterministic, such models cannot precisely describe the real phenomena, however give the possibility to allocate the most important factors of influence on the evolution and to define the nature of development of systems in global sense. If we take into account the probabilistic nature of all casual processes, we obtain considerably complicated mathematical models.

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Thus, there is no any contradiction since we investigate the deterministic mathematical models which have both the advantages and the disadvantages in comparison with stochastic models, which also have own advantages and disadvantages. Deterministic models give us the possibility to reveal and carry out the analysis of global tendencies of the evolution of systems by means of information technologies (the numerical calculation). In order to analyze stochastic models it is necessary to use the complicated mathematical tools, although we obtain the possibility to simulate perhaps only one historical event. So we have to pass between Scylla and Charybdis, and we can do it in such formulation of problems by means of computer modeling.

The advantage of mathematical modeling with application of the information technologies consists in the possibility of analyzing the influence of parameters and their various combinations on the evolution of systems, building evolutionary graphs and its analysis. Thus, we can obtain the evolutionary laws and avoid the negative development of systems [6, 7].

In the work we investigate the following three cases.

1) The arms race on the basis of the Richardson model. The purpose of States is own security. The limit cases are an infinite arms race or a mutual disarmament. The competing processes: the aggressive and peaceful evolution of the situation.
2) The war or battle on the basis of the modified Lanchester model. The purpose of States is a victory and gaining control over the opponent, the negative result is a defeat. The competing processes: the confrontation of armies of opponents.
3) The predator-prey model on the basis of the modified Lotka – Volterra model. The competing processes: the preys propagate in proportion to the population number, the number of the preys is reduced because of eating by predators, and the predators, on the contrary, propagate. If an amount of food in ecosystem is not enough, predators die out. The limit case of evolution of this system is the extinction of prey and predators, that is, dead zones are formed.

For the numerical analysis and building the graphs of solutions of the system of differential equations describing the studied models the mathematical Maple package is used. This package is one of the most reliable and enough simple tools for realization and visualization of any research where the mathematics is used. In this case the Runge – Kutta – Fehlberg method of 4(5) order is used (it is established by default).

II. The Richardson arms race model

A. *The formulation of the problem*

Let us study the first case when two States are in the status of confrontation. The Richardson arms race model is described by the system of the differential equations

$$\begin{cases} \dot{} \qquad x+r, \\ \dot{} \qquad +s. \end{cases} \qquad (1)$$

Here $x(t)$ and $y(t)$ define the expenses of the States on the arms, the values of the coefficients a and b describe how each State reacts to the increase or the decrease of armament of another State. It is clear that if one State increases the expenses on the development of science and industry connected with the military potential, then and another State will also increase the similar expenses. However, in this case the limiting process comes into the game – the population discontent: the demonstrations against the ruling regime, the pressure of the competing political parties upon the Government. This phenomenon is described by the coefficients m and n.

Each State being in confrontation seeks to show the own strength by means of demonstration of the modern arms, the foreign policy of the States can be aggressive or peace-loving. The positive and negative terms r and s describe a measure of peacefulness and eagerness to fight of the States. The solutions of the system are the functions $x(t)$ and $y(t)$ defined under the initial conditions x_0 and y_0.

The following 3 cases of the behavior of the Richardson model at $t \to \infty$ hold.

1) An infinite arms race: $x \to \infty$ and $y \to \infty$.

2) A mutual disarmament: $x \to 0$ and $y \to 0$.

3) An equilibrium of armament: $x \to \bar{x}$ and $y \to \bar{y}$. The equilibrium point $x \to \bar{x}$ and $y \to \bar{y}$ lies in the intersection of the straight lines

$$\begin{cases} ay - mx + r = 0, \\ bx - ny + s = 0. \end{cases}$$

B. *The analysis of the Richardson arms race model*

Let us consider four cases of combinations of parameters in the system of the differential equations describing the Richardson model.

Table I. Possible combinations of parameters in the Richardson model

1	$(mn - ab) > 0$	$r > 0$	$s > 0$
2	$(mn - ab) < 0$	$r > 0$	$s > 0$
3	$(mn - ab) > 0$	$r < 0$	$s < 0$
4	$(mn - ab) < 0$	$r < 0$	$s < 0$

In this table we take into account the more essential restriction on the arms race $(mn - ab) > 0$, the peacefulness or the aggression of foreign policy of the States and the opposite relationship between the coefficients.

Case 1. The situation when the States demonstrate the aggressive foreign policy is modeled. However, the population shows the active discontent with the arms race, the relationship of coefficients is $(mn - ab) = 0$. The values of parameters for which the numerical calculations were carried out are presented in Table II.

Table II. Case 1 – the initial conditions and the relationships between the parameters

a	b	m	n	r	s	x_0	y_0
1	1,5	2	1	0,5	1	30	15

The results of the numerical solution of system (1) are presented in the form of graphs in Fig. 1 (*a*). We see that the graphs intersect. It means that there exists the equilibrium point. Since at the initial moment of time the ratio $x_0/y_0=2$ holds, that is, the arms expenditures «*x*» is greater than the arms expenditures «*y*» in two times, the State «*x*» feels safe and cuts expenses on the arms, the second State «*y*», on the contrary, increases the arms race. As the result the graphs intersect at the point where the States are in equilibrium. Further, «*y*» continues to develop its military, ensuring the safety. However this process lasted for a short time since the State «*x*» is not enough aggressive and continues to cut the expenses on arms. What is the result of the evolution of the model: the weapons expenditures decrease for both States, the situation is stable, there is no an armed conflict.

Case 2. Further, similarly, we consider the States with the aggressive foreign policy, but with the relationship of coefficients $(mn-ab)<0$. The ratios of the values of the parameters at which the numerical calculations were carried out are given in Table III.

Table III. Case 2 – the initial conditions and the relationships between the parameters

a	*b*	*m*	*n*	*r*	*s*	x_0	y_0
2	1	1	1,5	1	1	0	1

The graphs illustrate the increase of the arms race for both States despite the existence of the equilibrium point (see Fig. 1 (*b*)). It is natural that the military conflict will be result of such evolution.

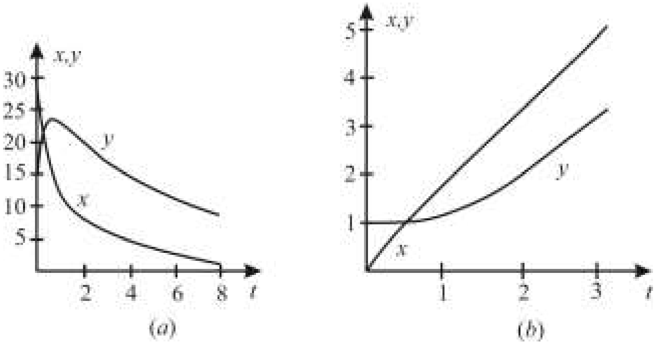

Fig. 1. The numerical solution of the Richardson model (case 1 and 2)

Case 3. The model when both States show the peace-loving foreign policy (see Table IV), the following relationship $(mn-ab)>0$ is true in this case.

Table IV. Case 3 – the initial conditions and the relationships between the parameters

a	*b*	*m*	*n*	*r*	*s*	x_0	y_0
1	1.5	2	1	–0.5	–1	30	15

The model is similar to the case 1 considered earlier. The difference consists in only that the values of the coefficients *r* and *s* are negative. Therefore, it is guaranteed there will be the disarmament of both States. As quickly it will occur, depends on the values of parameters and initial conditions.

Case 4. Let us consider now the situation when both States are peaceful and the inequality $(mn-ab)<0$ holds. The results of the numerical solution of system (1) are presented in the form of graphs in Fig. 2 (*a*) and (*b*). From the analysis of the results of calculations it follows that the initial conditions of the model (see Table V) define the evolution and the limit case of the arms race.

Table V. Case 4 – the initial conditions and the relationships between the parameters

a	*b*	*m*	*n*	*r*	*s*	x_0	y_0
2	1	1	1.5	–0.5	–1	6	6
2	1	1	1.5	–0.5	–1	2	4

The case, in which we consider the equal initial conditions for both States, is presented in Fig. 2 (*a*). The result of this model is an infinite arms race, and, therefore, an armed conflict. To reveal the influence of the initial conditions on the system evolution, we consider various values for x_0 and y_0, the other parameters are the same. The results of the numerical calculation are presented in the form of graphs (Fig. 2 (*b*)). Different values for the initial conditions lead to the mutual disarmament of the States. Thus, the presence of the friendly foreign policy of the States does not guarantee the peaceful outcome of the arms race, everything depends on the given initial conditions.

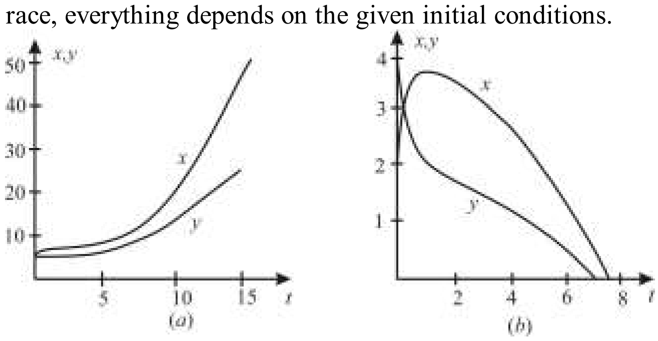

Fig. 2. The numerical solution of the Richardson model (case 4)

C. Conclusions

Let us draw the following conclusions.

1. If the restrictions on the arms race caused by discontent of the population in both States are enough large, $(mn-ab)>0$, then both in the case of aggressive and peace-loving foreign policy there is the disarmament of the States. In the case, when $r<0$ and $s<0$, we have the case of the mutual complete disarmament: $x \to 0$ and $y \to 0$, in the case, when $r>0$ and $s>0$, we have the stable situation. It says us about the importance of opposition to the arms race from the population and peace-loving foreign policy of the State.
2. If the restrictions on the arms race are small $(mn-ab)<0$, then the armed conflict can occur even in the presence of "friendly" foreign policy. However, if one of the States has the modern powerful weapon (i.e., its initial conditions far exceed the initial conditions of the opponent), then it feels safe, and the disarmament can occur at the peace-loving foreign policy. It means that if $r>0$, and $s>0$, then inevitably there is an unlimited growth of the arms race, and if $r<0$ and $s<0$, then the situation depends on the initial conditions. What does it speak about? The mankind can guarantee the lack of military actions actively opposing the arms race and aggressive policy of heads of States.

III. The modified Lanchester model

A. The formulation of the problem

The Lanchester model is closely connected with the Richardson model and represents the simplest model of fight of two opponents – two armies. In this model the state of the system is described by the point coordinates (x, y), $x > 0$, and $y > 0$, where x, y characterize the number of armies of the States – antagonists. For this model the system of the differential equations has the form

$$\begin{cases} \dot{} \; [illegible] \\ \dot{} \; [illegible] \end{cases} \quad (2)$$

where the weapon power of the army x is defined by the coefficient a, and the weapon power of the army y is b. In the system of equations it is put that the soldiers of the army x destroy the soldiers of the opponent army y with the rate a, and the soldiers of the army y destroy the soldiers of the opponent army x with the rate b. The Lanchester model is easily solved analytically, the solution is $ax^2 - by^2 = const$. It is clear that the integral curve is the hyperbola. The given initial conditions define the hyperbola branch of the system evolution. The written system of differential equations is very simple and cannot apply for describing the real process, although on the basis of the solution we can understand the limit behavior of the model. It is clear, that it is necessary to complicate the system. The goal of the study is not to introduce the functional dependence of the coefficients a and b [8], we investigate the influence of new terms in the right hand sides of the equations. In this case, modeling various situations of the course of war or battle, we can precisely define what term has the decisive influence on the result. We add the terms modeling simultaneous death of soldiers of both armies which are involved in the armed conflict. Such situation arises when both opponents use the biological and nuclear weapon. We consider extremely important influence of armies of the Allied Powers as well. Thus, we include into the system of equations such global terms which can define the course of evolution of mankind.

$$\begin{cases} [illegible] \beta xy + c, \\ [illegible] \varphi xy + e. \end{cases} \quad (3)$$

The terms $-\beta xy$ and $-\varphi xy$ in the system of equations define the extermination of soldiers of both armies. By means of the terms c and e we describe the entrance of Allies in war or battle. This model has the limit cases which are extremely important in modeling [9, 10]. It is obvious that at the temporizing policy of Allies, exactly the model Lotka – Volterra gives us the description of the competing processes. If there are no global phenomena of simultaneous extermination of soldiers, the system evolution is defined by the Lanchester model. In the offered model the arms power of armies of the States defined by the coefficients a and b influences on the values of the coefficients β and φ since the more powerful and modern weapon the army uses the more the soldiers die in war or battle. We will use the linear dependence which adequately describes this phenomenon: $\beta = 0.02a$ and $\varphi = 0.02b$.

B. The analysis of the Lanchester model

In this work we suppose that the dependence of the coefficients in the modified Lanchester system is linear. Let us consider two cases of the relations between the coefficients of system (3) presented in Table VI. If the coefficients are not equal, then we suppose that the values differ twice.

Table VI. The combinations of the coefficients for system (3)

1	$x_0 = y_0$	$c < e$	$a > b$
2	$x_0 < y_0$	$c < e$	$a > b$

Case 1. The values of the coefficients in the model are presented in Table VII. The strength of both armies is equal at the initial time; the advantage of the army «y» consists in that the strength of Allies are twice more; and the advantage of the army «x» is that the power of weapon is twice more. The results of numerical calculation are presented in the form of graphs (see Fig. 3).

Table VII. Case 1 – the initial conditions and the relationships between the parameters

x_0	y_0	a	b	β	φ	c	e
15	15	0.6	0.3	0.012	0.006	0.1	0.2

Though the initial conditions are identical for both States, the victory is won by the army «x» since it possesses the more powerful weapon. The influence of Allies is not so big to change the battle course. To turn the tide of the system evolution (the victory of the army «y»), the second state needs to increase the influence of allies by 15 times, that is, it is essential increasing (e = 3). Let us note that this number is received experimentally (see Fig. 3 (b)).

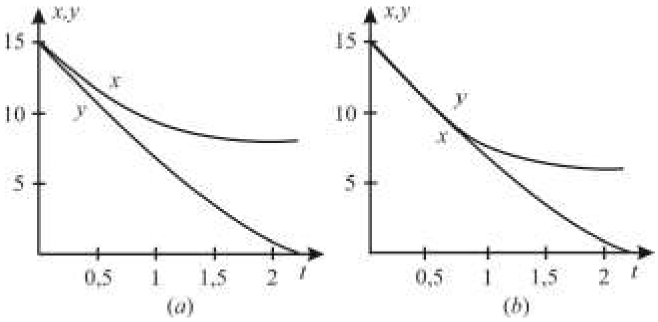

Fig. 3. The numerical solution of the modified Lanchester model (the strengths of both armies at the initial moment are equal)

Case 2. The situation when the strength of the army «y» is twice more than the strength of the army «x» is modeled. The values of the used parameters are given in Table VIII. Let us carry out the analysis of calculations on the basis of the graphs (Fig. 4).

Table VIII. Case 2 – the initial conditions and the relationships between the parameters

x_0	y_0	a	b	β	φ	c	e
4	8	0.6	0.3	0.012	0.006	0.1	0.2

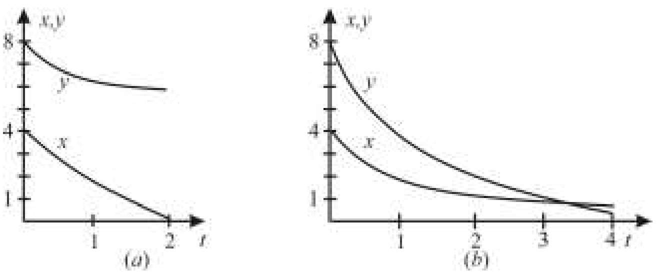

Fig. 4. The numerical solution of the modified Lanchester model ($a = 1.5$)

It seems obvious that the army «y» will destroy the army «x» due to the superiority in strength twice, and even its power of weapon which is also twice more, will not help to break the war course. What small States with the small size of army can do in this situation in order to resist the aggressor? Only the increase of the arms race provides the maximal security. From Fig. 4 (*b*) it follows that having increased the weapon power by 2.5 times ($a = 1.5$) the army «x» nevertheless will destroy the army of the opponent using the long-range high-precision technological weapons.

However, if the strength of Allies of the army «y» increases by 0.041, i.e., if the parameter $e = 0.241$, then the soldiers «x» also are defeated. The parameters used in this model are specified in Table IX. The results of modeling are presented in Fig. 5. The increase of the number of Allies also leads to the cardinal change of the result of military conflict.

Table IX. Taking into account the influence of Allies

x_0	y_0	a	b	β	φ	c	e
4	8	1.5	0.3	0.03	0.006	0.1	0.241

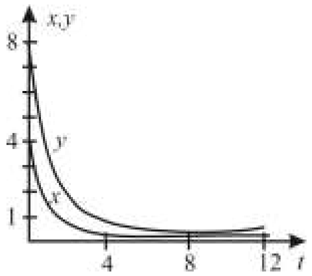

Fig. 5. The influence of Allies

C. The comparative analysis of the modified Lanchester model with the classical model

To separate the influence of the additional terms in the modified Lanchester model, we compare it with the classical model. It is natural to use in such analysis the equal initial conditions and the values of the coefficients a and b (Table X).

Table X. The initial conditions and the values of the coefficients for the classical and modified Lanchester model

x_0	y_0	a	b	β	φ	C	e
The classical model							
4	8	1.5	0.3				
The modified model							
4	8	1.5	0.3	0.03	0.006	0.1	0.2

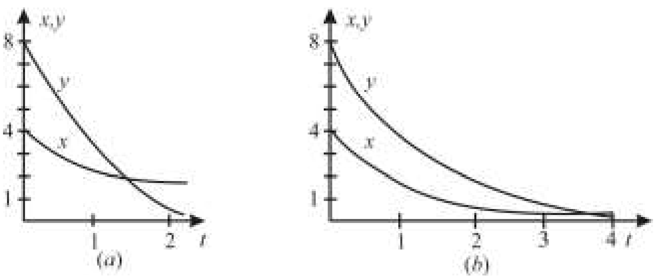

Fig. 6. The classical and modified Lanchester models – the result of battle

The results of numerical calculations for the classical (*a*) and modified (*b*) Lanchester models are presented in Fig. 6. From the analysis of the influence of the additional terms on the outcome of war or battle it follows that the army «x» wins the victory and destroys the soldiers of the army «y» in both cases since the values of the parameters taking into account the entrance of Allies into the conflict are small. However, naturally, in the modified model the transition to the phase of victory is carried out for a more long time.

Now we put the influence of the entrance of Allies into the military conflict in the coefficients a and b which determine the weapon power. We obtain the similar result, but now we are not able to understand any more what specifically affected on the battle course. In this case it is impossible to predict the result of the war between States, to allocate the influence of the arms race and Allies. The parameters at which modeling is carried out are specified in Table XI.

Table XI. The initial conditions and the values of coefficients for the classical Lanchester model

The initial conditions and parameters	x_0	y_0	a	b
The classical model	4	8	1.5	0.37

The deterministic modified Lanchester model which we study has the undoubted advantage in comparison with other models. In this work we take into account the most global and essential terms in the system of differential equations: the power of the high-precision and technological weapons; the entrance of Allies in the evolutionary game. Dynamic modelling allows us to define values and relationships of parameters which lead to the change of war or battle course at a certain time. It is obvious that if to consider a concrete historical event, we must take into account the probabilistic nature of random processes as well, to use the more complex mathematical tools, but in this work as it was already noted, we do not set such a goal.

IV. THE MODIFIED MODEL OF DYNAMICS OF THE PREY-PREDATOR SYSTEM

A. The formulation of the problem

Let us consider the classical predator-prey system, in this model there is no a competition between organisms of one species. If there are no "predators" in the system, then the "prey" population $x(t)$ propagates in proportion to its size with the increase rate $k > 0$, respectively and the number of predatory $y(t)$ without prey consumption die out in proportion to its number with the increase rate $l > 0$. The number of the prey is reduced because of the presence of predators, and the predators, on the contrary, propagate. This process is described by the well-known system of the differential equations

$$\begin{cases} \ldots) - ax(t)y(t), \\ \ldots t) + bx(t)y(t). \end{cases} \qquad (4)$$

Here the coefficient *a* defines the mortification of the prey when predators attack them, i.e., the predators and the prey interact in some sense, this term is directly proportional to the product *yx*. The coefficient *b* defines processing food in the form of the prey into the predator biomass, predators are breeding. If the biocenosis prospers, i.e., the biocenosis conditions are favorable, then the following two situations are possible: fluctuations of populations of predators and preys; the stable quantity of those and other species though the process of eating the preys by the predators is carried out continuously. However, the favorable conditions, in fact, are observed not always (for example, at forming dead zones). It is naturally that we need to consider some amendments to the classical model.

In this article we consider the model of the competing populations with the amendments p_1x^2 and p_2y^2 both for the prey and predators respectively:

$$\begin{cases} \ldots y + p_1x^2, \\ \ldots xy + p_2y^2. \end{cases} \qquad (5)$$

where the coefficients p_1 and p_2 define the influence of external factors on the system evolution of both positive and negative character. We consider the positive and negative signs, since a variety of conditions of the environment can be harmful or, on the contrary, bring a benefit to live organisms, promote survival or prevent to breed. The ecological factors are divided into: abiotic (the properties of inanimate nature), biotic (the types of influence of live organisms at each other) and anthropogenic (the human activity). To start the biological game the initial values for the prey x_0 and for the predators y_0 are given. The evolution of biological game and the final (in the case of formation of dead zones) depend on the signs and relationships of the corresponding coefficients. The terms defining the influence of separate components on the outcome of the biological game are written in the right hand sides of the differential equations. It is also important to consider the limit cases when dead zones are formed. The choice of the additional terms in this work is caused by the following reasons:

1) the nonlinear character of the reproduction rate of the prey;
2) the existence of competition between preys for food;
3) the nonlinear death of the prey due to the ecological catastrophes;
4) the nonlinear character of the rate of eating preys by predators;
5) the existence of the competition among predators for eating the prey;
6) the nonlinear character of the reproduction rate of predators.

Obviously, the considerable complication and increase of number of additional terms complicates understanding the evolution of the Lotka – Volterra system. In this work we consider the following global cases:

1) the model of equal ratio of the prey and predators due to reproduction of the prey and death of predators ($k=l$), and due to eating the prey by predators ($a = b$);
2) the model of the fast decrease of the prey abundance with respect to the increase of predator abundance due to eating the prey by predators.

Besides, the nonlinear additional terms influence on the system, and, perhaps, define the death of ecosystem. Let us note that the existence of nonlinearity in the system of differential equations allows us to take into account also the competitive exclusion principle (sometimes referred as Gause's law), since both the prey and predators compete fiercely between themselves for food and eating the prey respectively.

B. The model of equal ratio of the prey and predators due to reproduction of the prey and death of predators (k=l) and due to eating the prey by predators (a=b)

Let us consider the following combinations of values of parameters of system 5 (see Table XII): the equal ratio of the prey and predators due to reproduction of the prey and death of predators ($k = l$) and due to eating the prey by predators ($a = b$) taking into account the nonlinear character of the reproduction rate of the prey and predators (the sign +) and their death at ecological catastrophes and the existence of the competition for food (the sign –).

Table XII. The combinations of the values of the parameters for system (5)

1.	$k = l$	$a > b$	$p_1 > 0$	$p_2 > 0$
2.	$k = l$	$a > b$	$p_1 < 0$	$p_2 < 0$
3.	$k > l$	$a = b$	$p_1 > 0$	$p_2 > 0$
4.	$k > l$	$a = b$	$p_1 < 0$	$p_2 > 0$
5.	$k > l$	$a = b$	$p_1 < 0$	$p_2 < 0$
6.	$k < l$	$a = b$	$p_1 > 0$	$p_2 < 0$
7.	$k < l$	$a = b$	$p_1 < 0$	$p_2 < 0$

Case 1. Let us consider the following situation: the coefficients of the increase rate of the prey and the death of the predators are equal; the living conditions for predator and prey population are favorable, the initial conditions – the abundance of predators are twice less than the the abundance of preys (see Table XIII).

Table XIII. The combinations of the values of the parameters for case 1

x_0	y_0	a	b	k	l	p_1	p_2
100	50	0.04	0.02	0.3	0.3	0.0004	0.0004

The results of calculations are presented in Fig. 7 (*a*). The behavior of the system practically coincides with the behavior of the classical Lotka – Volterra system. From the figure one can see that even at the positive influence of the external environment promoting the nonlinear reproduction, the population can restore its abundance not at once. This process has the periodic character.

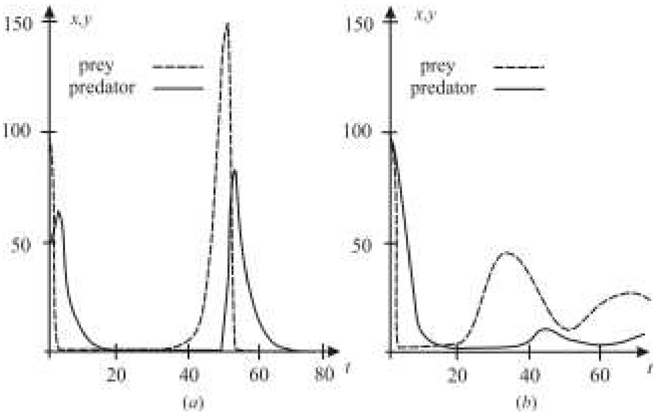

Fig. 7. The dynamics of the population number, case (1):
(*a*) – the conditions of the biocenosis are favorable;
(*b*) – the conditions of the biocenosis are adverse

Case 2. The situation at which the coefficients of reproduction rate of the prey and death rate of the predators are equal, the conditions of activity exert the negative influence on the development of populations both for the prey and for the predators, $p_1 = p_2 = -0.006$ (see Table XIII).

From Fig. 7 (*b*) one can see that such influence leads to the damped oscillations in time, the number of species decreases. Let us note that at the initial time the behavior of the modified model coincides with the behavior of the classical Lotka – Volterra model. In time the increase of the number of the prey population corresponds to the classical model. Therefore, even the insignificant negative effect on the biocenosis leads to the reduction of the number both for the prey and predators.

Case 3. The situation when the reproduction rate of the prey is higher in comparison with the mortality rate of the predators. The conditions of environment provide the positive development of both populations (see Table XIV).

Table XIV. The combinations of the values of the parameters for case 3

x_0	y_0	a	b	k	l	p_1	p_2
100	100	0.02	0.02	0.7	0.3	0.0006	0.0006

One can see (Fig. 8 (*a*)) that the number of the prey is rapidly going down. It is connected with the sharp increase of the number of the predatory species. After decreasing the prey population is restored for a long time that leads to the sharp decrease of the number of the predators which suffer a lack of food. While predators die out, the preys begin to breed actively. It means that we observe the oscillatory process.

Case 4. The situation when the increase rate of the prey is greater than the mortality rate of the predators. The conditions are favorable only for the last, the prey suffers a crisis $p_1 = -0.006, p_2 = 0.006$. The values x_0, y_0 are equal for both species at the initial moment of time. The results of calculations for this case are presented in Fig. 8 (*b*).

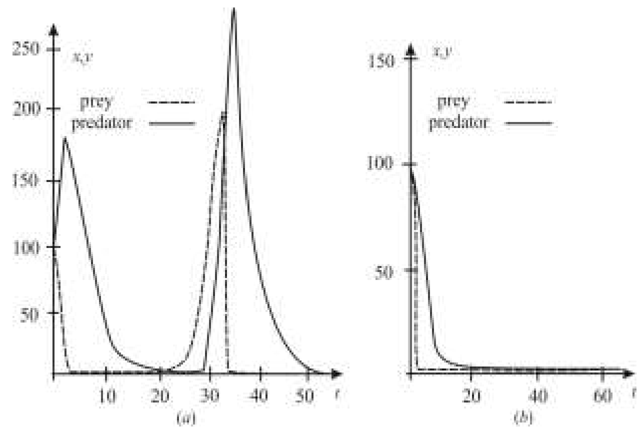

Fig. 8. The dynamics of number of population of system (5):
(*a*) – the conditions of the biocenosis are favorable;
(*b*) – the conditions of the biocenosis are adverse for the prey

We see that the prey stops to propagate quickly enough, their number reaches zero. The population of predators also dies out but more slowly. The populations do not have enough time to develop, the so-called dead zone is formed.

Case 5. The situation when the increase rate of the prey is greater than the mortality rate of the predatory species $k = 0.8, l = 0.6$. The conditions of the environment are adverse both for the prey and for the predators $p_1 = p_2 = -0.006$, the initial conditions are the same.

From Fig. 9 (*a*) we see that the nature of the system evolution has the oscillatory character, due to the higher birth rate of prey the system survives but the amplitude of fluctuations falls because of adverse conditions. What to do in this uneasy situation? Of course – to improve the living conditions for the prey, perhaps in this case, the ecosystem will survive.

Case 6. The situation when the increase rate of the prey is less than the mortality rate of the predatory species $k = 0.2, l = 0.6$. The conditions of the environment are favorable both for the prey and for the predators $p_1 = 0.0006, p_2 = 0.0008$, the initial conditions are the same. From Fig. 9(*b*) we can see that at the beginning the abundance of the prey sharply decreases, the predators actively propagate. Further, the abundance of the prey and predator populations decreases since the prey becomes less, and the predators have nothing to eat. The mortality rate is higher than the birth rate that negatively influences on the system. The populations are not able to continue the existence, despite the favorable conditions. Thus, in this case it is necessary to influence on the increase of the birth rate of the prey.

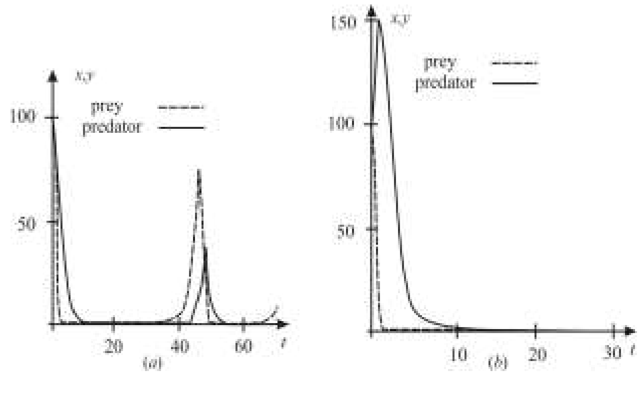

Fig. 9. The dynamics of the population abundance of system (5): (*a*) – the conditions of the biocenosis are adverse; (b) – the conditions of the biocenosis are favorable

Case 7. Here we consider the similar case but under the negative influence of the additional terms $p_1 = -0.006,\ p_2 = -0.008$, $k = 0.4, l = 0.6$. In this case, despite the increase of the birth rate of the prey, both species also die, and practically at once both the prey and predators die, since the negative factors influence on the ecosystem.

Thus:

1. The influence of the additional factors on the biocenosis evolution is essential, the positive values of coefficients cause the biocenosis prosperity, the negative values lead to the death both of the prey and predators. In order to control a real biocenosis and not to allow forming dead zones, it is necessary, at least, to promote considerable reproduction of the prey that leads respectively to the increase of the abundance of predators.
2. Only enough big reproduction rate of the prey which, besides, must be greater in comparison with the extinction rate of predators, can overcome the nonlinear death of species caused by ecological catastrophes.

C. The model of fast decrease of the prey abundance with respect to the increase of the predator abundance due to eating the prey by predators

We consider the cases when ($a/b > 1$), i.e., the coefficient of murder of the prey is greater than the transformation coefficient of biomass of the prey into the biomass of the predators, and ($a/b < 1$), i.e., the predators quickly breed due to eating the preys, their number quickly increases. Besides, the nonlinear effects can change the course of the evolution of system. Let us simulate the cases of using the combinations of the values of the parameters for system (5) inTable XV.

Table XV. The combinations of the values of the parameters for system (5)

1.	$k = l$	$a/b>1$	$p_1 > 0$	$p_2 > 0$
2.	$k = l$	$a/b>1$	$p_1 < 0$	$p_2 < 0$
3.	$k < l$	$a/b>1$	$p_1 > 0$	$p_2 > 0$
4.	$k < l$	$a/b>1$	$p_1 < 0$	$p_2 < 0$
5.	$k > l$	$a/b<1$	$p_1 > 0$	$p_2 > 0$
6.	$k < l$	$a/b<1$	$p_1 < 0$	$p_2 < 0$

Case 1. The situation when the reproduction rate of the prey and the mortality rate of the predators are equal, the environment conditions are favorable. The number of the prey sharply decreases during the attack of predators that can be caused by the aggression due to the nonlinear nature of reproduction (see Table XVI). In Fig. 10 at the beginning we see the growth of the abundance of predators, the prey very quickly perishes. Further, we observe the extinction of populations, despite the positive coefficients of the nonlinear terms. If to consider the negative coefficients of the nonlinear terms, then the death of populations begins at the initial moment of time (case 2). Further, the nature of evolution is similar.

Table XVI. The combinations of the values of the parameters for case 1

x_0	y_0	a	b	k	l	p_1	p_2
100	100	0.5	0.3	0.2	0.2	0.0003	0.0005

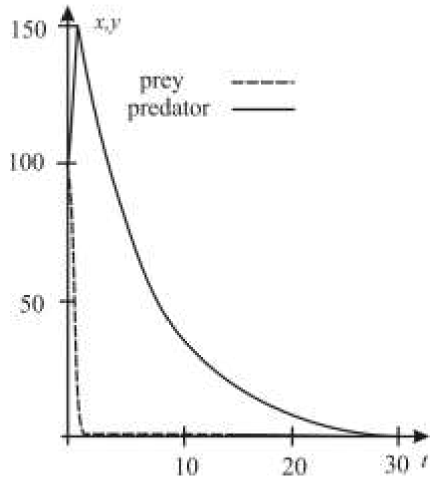

Fig. 10. The dynamics of the population number of system (5): the conditions of the biocenosis are favorable

Case 3. The situation when the birth rate of the prey is lower in comparison with the death rate of the predators, the conditions are favorable for both species (see Table XVII).

Table XVII. The combinations of the values of the parameters for case 3

x_0	y_0	a	b	k	l	p_1	p_2
100	50	0.5	0.3	0.02	0.04	0.0004	0.0003
						–0.0004	–0.0003

In Fig. 11 the results of calculations are presented. The number of the prey sharply declines up to zero. At the beginning the predators actively breed due to the favorable conditions and further they continue to survive (the life expectancy increases) long enough, their number decreases slowly that considerably differs from the cases considered earlier.

Case 4. The difference of this case is only that the conditions are adverse (see Table XVI) therefore the number decreases up to zero very quickly.

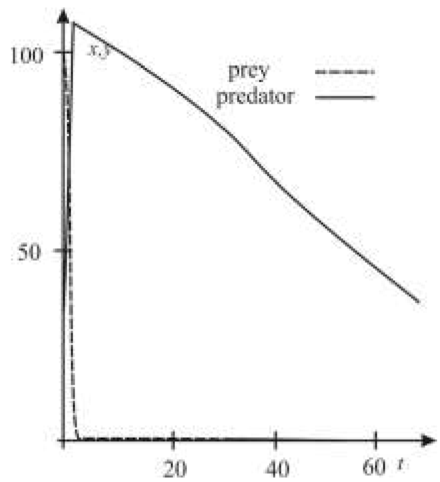

Fig. 11. The dynamics of the population number of system (5): the conditions of the biocenosis are favorable, the birth rate of the prey is low

Case 5. The situation when the reproduction rate of the prey is much greater than the mortality rate of the predators. The conditions of the environment correspond to the favorable nonlinear reproduction for both species (see Table XVIII).

Table XVIII. The combinations of the values of the parameters for case 5

x_0	y_0	a	b	k	l	p_1	p_2
100	50	0.2	0.3	0.4	0.2	0.0005	0.0007

In this model we take into account that the number of the deceased predators is sharply reduced in comparison with their gain due to plentiful eating the prey as well. In Fig. 12 we see that at the first time the predators actively breed but because of low birth rate of the prey and the shortage of food they perish despite the favorable conditions.

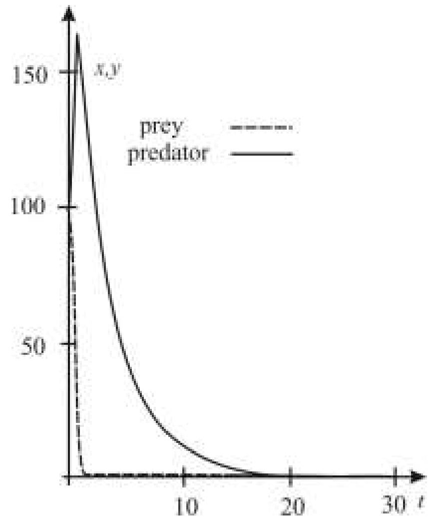

Fig. 12. The dynamics of the population number of system (5): the conditions of the biocenosis are favorable

Case 6. The situation when the growth rate of the number of the preys decreases in comparison with the mortality rate of the predators. The nonlinear additional terms are negative. In this model we take into account that the deceased predators is sharply reduced in comparison with their gain due to plentiful eating the preys as well (see table XIX).

Table XIX. The combinations of the values of the parameters for case 6

x_0	y_0	a	b	k	l	p_1	p_2
100	50	0.3	0.6	0.4	0.8	–0.004	–0.007

In this case we also observe the sharp decrease of quantity of both species, even the possibility of plentiful eating cannot overcome the low birth rate of the prey, the high mortality rate of the predators and the negative influence of the nonlinear factors.

V. Conclusion

In order that the confrontation between the States did not lead to the beginning of war it is necessary to take into account the following conditions in Richardson's and Lanchester's models:

1) the population activity of the States for modeling the restriction of the arms race;
2) the peace-loving foreign policy;
3) the presence of strong Allies in the case of the war beginning.

The interactive numerical analysis (the possibility to change the parameters) gives us the possibility to visualize the results and to show the current situation. On its basis we can do the forecasts for the future and prevent rather a real catastrophe – the extermination of mankind.

The studied modified predator-prey model describes the system evolution with the competing processes and nonlinear influence of both external and internal factors. The analysis of the obtained results allows us to predict and, therefore, to avoid creation of dead zones in the nature. The advantage of this model consists in that the nonlinear terms describe practically all set of possible cases of both the positive and negative influence of the environment. Various combinations of parameters of system of differential equations define the biocenosis evolution.

Let us draw the conclusions on the basis of the carried out numerical calculations and the analysis of results of the solution of the modified Lotka – Volterra system with various parameters.

The following conditions are necessary for survival of an ecosystem:

1) enough big the reproduction rate of the prey;
2) the reproduction rate of the prey must be greater than the mortality rate of predators;
3) the existence of positive nonlinear character of reproduction rate of the prey.

Only in this case we observe the oscillatory nature of evolution of a biocenosis.

In the case of ecological catastrophes the biocenosis inevitably perishes. For its restoration the reproduction rate of the prey needs to be significantly increased. Then the ecosystem restoration is possible even at insignificant negative nonlinear terms.

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